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# Extreme points of an intersection of operator intervals(Nonlinear Analysis and Convex Analysis)

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## Extreme points of an intersection of operator intervals

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Though the title of my talk at the RIMS meeting was *Extreme points of an intersection of matrix intervals*, here the problem is treated in the Hilbert space setting with title *Extreme points of an intersection of operator intervals*. The detail with full proof will appear in the **Proceedings of International Mathematics Conference '94, Kaohsiung, Taiwan**, World Scientific, 1996.

### 1. Introduction

Let  $\mathcal{H}$  be a (complex) Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ , and  $\mathcal{B}(\mathcal{H})$  the space of (bounded linear) operators on  $\mathcal{H}$ . When  $\dim(\mathcal{H}) = n < \infty$ , with respect to a suitably chosen orthonormal basis,  $\mathcal{B}(\mathcal{H})$  is identified with the space of  $n \times n$  matrices.

Recall that an operator  $A$  is *selfadjoint* if  $A^* = A$  where  $A^*$  is the *adjoint* of  $A$ , that is,  $\langle A^*x, y \rangle = \langle x, Ay \rangle$  ( $x, y \in \mathcal{H}$ ). Selfadjointness of  $A$  is characterized by that  $\langle x, Ax \rangle$  is real for all  $x \in \mathcal{H}$ . Recall further that  $A$  is *positive (semidefinite)* if  $\langle x, Ax \rangle \geq 0$  for all  $x \in \mathcal{H}$ . Order relation  $A \geq B$  for a pair of selfadjoint operators  $A, B$  is defined as  $A - B$  is positive. Therefore  $A \geq 0$  means positivity of  $A$ . We write  $A > B$  to mean that  $A \geq B$  and  $A - B$  is invertible, which is equivalent to  $A - B \geq \varepsilon I$  for some  $\varepsilon > 0$  where  $I$  is the identity operator.

Given  $A_1, A_2, \dots, A_m \geq 0$ , let us observe the set

$$\Delta(A_1, A_2, \dots, A_m) \stackrel{\text{def}}{=} \{X \mid 0 \leq X \leq A_j \text{ } j = 1, 2, \dots, m\}.$$

In particular,  $\Delta(A)$  for  $A \geq 0$  is the *operator interval*  $\{X \mid 0 \leq X \leq A\}$ , so that  $\Delta(A_1, A_2, \dots, A_m)$  is an intersection of operator intervals;

$$\Delta(A_1, A_2, \dots, A_m) = \bigcap_{j=1}^m \Delta(A_j).$$

Since  $\Delta \equiv \Delta(A_1, A_2, \dots, A_m)$  is a convex set, compact with respect to the weak operator topology, according to the Krein-Milman theorem it has plenty many extreme points. Recall that  $X \in \Delta$  is *extreme* if  $X = (Y + Z)/2$  for  $Y, Z \in \Delta$  is possible only when  $Y = Z = X$ . Let us denote by  $\text{ex}\Delta \equiv \text{ex}\Delta(A_1, A_2, \dots, A_m)$  the set of extreme points of  $\Delta$ .

Our aim is a detailed study of the extreme points of  $\Delta$ . In Section 3 we give several necessary and sufficient conditions for  $X \in \Delta$  to be an extreme point. In Section 4 we present labelling of extreme points and in Section 5, under the assumption  $\dim(\mathcal{H}) < \infty$  and  $m = 2$ , we give a complete parametrization of extreme points of  $\Delta$ . The final part, Section 6, contains an algorithm of construction of an extreme point, associated with arbitrarily given  $X \in \Delta$ . In the preliminary part, Section 2, we recall various properties of two basic operations, necessary for our study; parallel sum and short.

This paper enlarges and extends the content of an unpublished manuscript [Ao 89], a part of which has been made public in [P 91].

## 2. Preliminaries

Given  $A, B > 0$ , the operator

$$A : B \stackrel{\text{def}}{=} (A^{-1} + B^{-1})^{-1}$$

is called the *parallel sum* of  $A$  and  $B$ . This notion was first introduced in [AD 69] as mathematical description of the impedance of parallel connection of electrical networks. From the standpoint of quadratic forms the following variational description is more useful (see [AT 75]) and shows that parallel addition corresponds to the so-called *inf-convolution* of two positive quadratic forms (see [M 88]);

$$\langle x, (A : B)x \rangle = \inf \{ \langle y, Ay \rangle + \langle z, Bz \rangle ; y + z = x \} \quad (x \in \mathcal{H}). \quad (2.1)$$

Via (2.1) we can extend the definition of parallel sum to any pair of positive operators. This extended operation enjoys the following properties (see [AT 75], [NA 76], [PS 76] and [EL 89]);

$$A, B \geq A : B = B : A, \quad (2.2)$$

$$(A : B) : C = A : (B : C), \quad (2.3)$$

$$(\lambda A) : (\mu A) = \frac{\lambda\mu}{\lambda + \mu} A \quad (\lambda, \mu > 0), \quad (2.4)$$

$$A_1 \geq A_2, B_1 \geq B_2 \implies A_1 : B_1 \geq A_2 : B_2, \quad (2.5)$$

and more generally

$$A_k \downarrow A, B_k \downarrow B \implies A_k : B_k \downarrow A : B, \quad (2.6)$$

where  $A_k \downarrow A$  say, means that  $A_k$  decreasingly converges to  $A$  in strong operator-topology as  $k \rightarrow \infty$ , and

$$(S^*AS) : (S^*BS) = S^*(A : B)S \quad \text{for invertible } S \in \mathcal{B}(\mathcal{H}). \quad (2.7)$$

In view of commutativity (2.2) and associativity (2.3) there is no confusion to use

$$\prod_{j=1}^m : A_j \equiv (\dots((A_1 : A_2) : A_3) \cdots : A_m).$$

Another notion we need is short (operation). For  $A, X \geq 0$  the sequence  $(kX) : A$  increases as  $k \rightarrow \infty$ . Following [Ao 76] let us define  $[X]A$  as its strong limit;

$$[X]A \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} (kX) : A. \quad (2.8)$$

By (2.5) the following are clear from definition (2.8):

$$A \geq [X]A, \quad (2.9)$$

$$A_1 \geq A_2 \geq 0, X_1 \geq X_2 \implies [X_1]A_1 \geq [X_2]A_2, \quad (2.10)$$

and by (2.3) and (2.4)

$$\alpha X_1 \geq X_2 \geq \beta X_1 \text{ for some } \alpha, \beta > 0 \implies [X_1]A = [X_2]A. \quad (2.11)$$

Every positive operator  $X$  admits a unique *positive square-root*  $X^{1/2}$ , that is,  $X^{1/2} \geq 0$  and  $(X^{1/2})^2 = X$ . For general  $X \in \mathcal{B}(\mathcal{H})$  the positive square root of  $X^*X$  is called the *modulus* of  $X$  and denoted by  $|X|$ . When  $X$  is selfadjoint,  $|X|$  commutes with  $X$ .

We use  $\text{ran}(X)$  and  $\ker(X)$  to denote the *range* and the *kernel* of  $X \in \mathcal{B}(\mathcal{H})$  respectively. Then obviously

$$\text{ran}(X)^\perp = \ker(X^*) \quad \text{and} \quad \ker(X)^\perp = \overline{\text{ran}(X^*)}, \quad (2.12)$$

where  $\{\}^\perp$  denotes the orthogonal complement. For  $X \geq 0$  we have

$$\ker(X) = \ker(X^{1/2}) \quad \text{and} \quad \text{ran}(X) \subset \text{ran}(X^{1/2}) \subset \overline{\text{ran}(X)}. \quad (2.13)$$

When  $\dim(\mathcal{H}) < \infty$ , every subspace is closed, so that positive  $A$  is invertible if  $\text{ran}(A) = \mathcal{H}$ , or equivalently  $\ker(A) = \{0\}$ .

For  $A \geq 0$  and selfadjoint  $X \in \mathcal{B}(\mathcal{H})$

$$A \geq X \geq -A \iff X = A^{1/2}CA^{1/2}, \quad (2.14)$$

where  $C$  is a (selfadjoint) *contraction*, that is,  $I \geq C \geq -I$ . If in addition  $\text{ran}(C) \subset \overline{\text{ran}(A)}$  is required,  $C$  is uniquely determined. In a similar way  $A \geq X \geq 0$  is characterized by  $I \geq C \geq 0$  in (2.14).

Range inclusion is characterized by an operator inequality (see [D 66]); for  $X, Y \in \mathcal{B}(\mathcal{H})$

$$\text{ran}(X) \supset \text{ran}(Y) \iff \alpha XX^* \geq YY^* \quad \text{for some } \alpha \geq 0. \quad (2.15)$$

A consequence of (2.14) is that the operation  $A \mapsto [X]A$  is determined only by the range space  $\mathcal{L} = \text{ran}(X^{1/2})$ . Therefore  $[X]A$  will be called the *short* of  $A$  to the operator range  $\mathcal{L}$ .

If  $0 \leq C, D \leq I$  and  $\text{ran}(C), \text{ran}(D) \subset \overline{\text{ran}(A)}$  then we have

$$(A^{1/2}CA^{1/2}) : (A^{1/2}DA^{1/2}) = A^{1/2}(C : D)A^{1/2}. \quad (2.16)$$

(see [PS 76]).

If  $\text{ran}(X)$  is closed, then by (2.13)  $[X]A = [P]A$  where  $P$  is the projection to  $\text{ran}(X) = \text{ran}(X^{1/2})$ . The shorted operator  $[P]A$  is written in the block operator

matrix with respect to the decomposition  $\mathcal{H} = \text{ran}(P) \oplus \text{ker}(P)$  in the following form (see [A 71]):

$$[P]A = \begin{bmatrix} A_{11} - A_{12} \cdot A_{22}^{-1} \cdot A_{21} & 0 \\ 0 & 0 \end{bmatrix} \quad (2.17)$$

for

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \geq 0.$$

Here we remark that positivity of  $A$  ensures that there is an operator  $K$  from  $\text{ker}(P)$  to  $\text{ran}(P)$  such that  $\|K\| \leq 1$  and  $A_{12} = A_{11}^{1/2} K A_{22}^{1/2}$ , and  $A_{12} \cdot A_{22}^{-1} \cdot A_{21}$  should be understood as

$$A_{12} \cdot A_{22}^{-1} \cdot A_{21} \stackrel{\text{def}}{=} A_{11}^{1/2} K K^* A_{11}^{1/2}.$$

The shorted operator  $[P]A$  for a projection  $P$  admits the following maximum characterization, originally due to M. Krein (see [AT 75]);

$$[P]A = \max\{Y \mid 0 \leq Y \leq A, \text{ran}(Y) \subset \text{ran}(P)\}. \quad (2.18)$$

For general  $X \geq 0$  we have

$$[X]A = A^{1/2} Q A^{1/2}, \quad (2.19)$$

where  $Q$  is the projection to the closure of  $A^{-1/2} \text{ran}(X) \equiv \{x \in \mathcal{H} \mid A^{1/2} x \in \text{ran}(X)\}$  (see [P 78] and [K 84]). Conversely for any projection  $Q$  with  $\text{ran}(Q) \subset \overline{\text{ran}(A)}$

$$A^{1/2} Q A^{1/2} = [X]A \quad \text{with } X = A^{1/2} Q A^{1/2}. \quad (2.20)$$

It is known (see [Ao 76], [P 78] and [EL 86]) that

$$[X]A = A \iff A = \sup\{Y \mid 0 \leq Y \leq A \text{ and } Y \leq \lambda X \text{ for some } \lambda = \lambda(Y) \geq 0\} \quad (2.21)$$

Sum and intersection of operator ranges are well determined (see [FW 71]): for  $A, B \geq 0$

$$\text{ran}(A) + \text{ran}(B) = \text{ran}(A^2 + B^2)^{1/2}, \quad (2.22)$$

and

$$\text{ran}(A) \cap \text{ran}(B) = \text{ran}(A^2 : B^2)^{1/2}. \quad (2.23)$$

The range of a short to a closed subspace is also well determined (see [AT 75]): for projection  $P$

$$\text{ran}([P]A)^{1/2} = \text{ran}(P) \cap \text{ran}(A^{1/2}). \quad (2.24)$$

It follows from (2.8) and (2.24) that for  $A, B \geq 0$

$$\text{ran}(A^{1/2}) \cap \text{ran}(B^{1/2}) = \{0\} \iff A : B = 0. \quad (2.25)$$

### 3. Characterizations

It is well known that  $ex\Delta(I)$  for identity operator  $I$  consists of all projections (see [Sa 71, p.12]). Remark that a projection  $X$  is characterized as a selfadjoint idempotent, that is,  $X(I-X) = 0$ , and also that when  $I \geq X \geq 0$  then  $X(I-X) = X : (I-X)$ . By definition (2.8)  $[X]I$  for every  $X \geq 0$  coincides with the projection to  $\overline{ran(X)}$ . These together can be formulated as various characterizations of  $ex\Delta(I)$  in the following Lemma (see [AT 88] and [EL 86]):

**Lemma 1.**

$$\begin{aligned} ex\Delta(I) &= \{P \mid \text{projection } P\} \\ &= \{[X]I \mid X \geq 0\} \\ &= \{X \in \Delta(I) \mid X : (I-X) = 0\}. \end{aligned}$$

An easy consequence is that for a projection  $P$

$$ex\Delta(P) = \{Q \mid \text{projection } Q \leq P\}.$$

Given  $A \geq 0$  let  $P$  be the projection to  $\overline{ran(A)}$ . Then by (2.14) the affine map  $X \mapsto A^{1/2}XA^{1/2}$  transforms bijectively  $\Delta(P)$  to  $\Delta(A)$ , so that the affine structure of  $ex\Delta(A)$  can be copied from Lemma 1 and its consequence via (2.16), by using (2.19) and (2.20). Therefore we have a complete answer for the case  $m = 1$  (cf. [AT 88], [EL 86], [AMT 92] and [P 92]).

**Theorem 2.** For  $A \geq 0$

$$\begin{aligned} ex\Delta(A) &= \{A^{1/2}PA^{1/2} \mid \text{projection } P \text{ such that } ran(P) \subset \overline{ran(A)}\} \\ &= \{[X]A \mid X \geq 0\} \\ &= \{X \in \Delta(A) \mid X : (A-X) = 0\}. \end{aligned}$$

Remark that, in general, not every extreme point of  $\Delta(A)$  is of the form  $[P]A$  for a projection  $P$  (see [P 91] and [GM 94]).

The case  $m > 1$  is more delicate, but still can be derived from Lemma 1 (cf. [Ao 89] and [P 91])

**Theorem 3.** Let  $A_j \geq 0$  ( $j = 1, 2, \dots, m$ ) and  $X \in \Delta(A_1, \dots, A_m)$ . Then the following conditions on  $X$  are mutually equivalent.

- (i)  $X \in ex\Delta(A_1, \dots, A_m)$ .
- (ii)  $X : \{\prod_{j=1}^m (A_j - X)\} = 0$ .
- (iii)  $ran(X^{1/2}) \cap \bigcap_{j=1}^m ran(A_j - X)^{1/2} = 0$ .

Also each of (i), (ii) and (iii) is equivalent to any of the following.

- (i')  $X \in ex\Delta([X]A_1, \dots, [X]A_m)$ .

$$(ii') \quad X : \{\prod_{j=1}^m : ([X]A_j - X)\} = 0.$$

$$(iii') \quad \text{ran}(X^{1/2}) \cap \bigcap_{j=1}^m \text{ran}([X]A_j - X)^{1/2} = 0.$$

**Theorem 4.** *Let  $A_j \geq 0$  ( $j = 1, 2, \dots, m$ ) and  $X \in \Delta(A_1, \dots, A_m)$ . Then each of the following conditions implies any one (and all) in Theorem 3.*

$$(iv) \quad \prod_{j=1}^m : ([X]A_j - X) = 0.$$

$$(v) \quad \ker(X) + \sum_{j=1}^m \ker(A_j - X) = \mathcal{H}.$$

(vi) *There are mutually commuting (not necessarily selfadjoint) idempotents  $Q_j \in \mathcal{B}(\mathcal{H})$  ( $j = 0, 1, 2, \dots, m$ ) such that*

$$Q_i Q_j = \delta_{ij} Q_j \quad (i, j = 0, 1, 2, \dots, m), \quad \sum_{j=0}^m Q_j = I, \quad \text{and} \quad X = \sum_{j=1}^m A_j Q_j.$$

When  $\dim(\mathcal{H}) < \infty$ , each of (iv), (v) and (v) is equivalent to any one (and all) in Theorem 3.

#### 4. Labelling

All characterizations of extremality in Theorem 3 and Theorem 4 are of qualitative nature except (vi). Representation (vi), however, has a defect in using non-selfadjoint summands. In fact, there is no guarantee of selfadjointness of  $A_j Q_j$  ( $j = 1, 2, \dots, m$ ).

In the finite dimensional case, however, the following theorem gives a labelling of every extreme point  $X$  of  $\Delta(A_1, A_2, \dots, A_m)$  in terms of those in  $\Delta(Y_j)$  ( $j = 1, 2, \dots, m$ ) where  $Y_j$  ( $j = 1, 2, \dots, m$ ) are defined recursively from  $X$  (cf. [Ao 89] and [P 91]).

**Theorem 5.** *Let  $A_j \geq 0$  ( $j = 1, 2, \dots, m$ ) and  $X \in \Delta(A_1, \dots, A_m)$ . Then if there are  $X_k \geq 0$  ( $k = 1, 2, \dots, m$ ) such that*

$$X = \sum_{j=1}^m X_j \quad \text{and} \quad X_k \in \text{ex}\Delta(A_k - \sum_{j=1}^{k-1} X_j) \quad (k = 1, 2, \dots, m), \quad (4.1)$$

where  $\sum_{j=1}^{k-1} X_j \equiv 0$  for  $k = 1$ , then  $X$  is an extreme point of  $\Delta(A_1, \dots, A_m)$ .

Those  $X_k$  ( $k = 1, 2, \dots, m$ ), satisfying (4.1) and the additional condition that for some  $1 > \varepsilon > 0$

$$(1 - \varepsilon)(A_k - \sum_{j=1}^k X_j) \geq \sum_{j=k+1}^m X_j \equiv X - \sum_{j=1}^k X_j \quad (k = 1, 2, \dots, m) \quad (4.2)$$

are unique for  $X$  if exist.

When  $\dim(\mathcal{H}) < \infty$ , existence of those  $X_k$  ( $k = 1, 2, \dots, m$ ), satisfying (4.1) and (4.2), is always guaranteed for every  $X \in \text{ex}\Delta(A_1, A_2, \dots, A_m)$ .

## 5. Parametrization

When  $\dim(\mathcal{H}) < \infty$  and  $m = 2$ , in Theorem 5  $X_1$  is considered as a free parameter for  $X \in \text{ex}\Delta(A_1, A_2)$  because  $X_2 = X - X_1$ . But the requirements  $(1 - \varepsilon)(A_1 - X_1) \geq X - X_1$  for some  $0 < \varepsilon < 1$  and  $X_2 \in \text{ex}\Delta(A_2 - X_1)$  are still restrictive. In this section, using a result from the theory of indefinite inner product spaces, we shall present a more transparent parametrization of  $\text{ex}\Delta(A_1, A_2)$  along an idea of [Ao 93].

Let  $\dim(\mathcal{H}) < \infty$  and  $m = 2$ . For brevity let us write

$$A \stackrel{\text{def}}{=} A_1 \quad \text{and} \quad B \stackrel{\text{def}}{=} A_2.$$

According to Theorem 4 extremity of  $X \in \Delta(A, B)$  is characterized by

$$([X]A - X) : ([X]B - X) = 0.$$

When restricted to  $\text{ran}(X) = \text{ran}(X^{1/2})$ , each of  $[X]A$ ,  $[X]B$  and  $X$  is positive invertible by (2.26). Therefore assuming  $A, B > 0$  we shall consider parametrization of all invertible extreme points of  $\Delta(A, B)$ . In this case requirement for extremity of  $X$  becomes

$$(A - X) : (B - X) = 0. \quad (5.1)$$

Let  $A, B > 0$ , and

$$C \stackrel{\text{def}}{=} \frac{1}{2}(A - B).$$

If  $A \geq B$  or  $A \leq B$  then  $\Delta(A, B)$  reduces to either  $\Delta(B)$  or  $\Delta(A)$ , and a parametrization of  $\Delta(A, B)$  is already known by Theorem 2. Therefore we shall assume that  $C$  is indefinite, that is,  $C \not\geq 0$ , and  $C \not\leq 0$ . The space  $\mathcal{H}$  is written as an orthogonal sum

$$\mathcal{H} = \text{ran}(C) \oplus \ker(C). \quad (5.2)$$

Since

$$\text{ran}(C) = \text{ran}(|C| - C) \oplus \text{ran}(|C| + C),$$

we can also write

$$\mathcal{H} = \text{ran}(|C| - C) \oplus \text{ran}(|C| + C) \oplus \ker(C). \quad (5.3)$$

Let

$$n_+ = \dim(|C| - C) \quad \text{and} \quad n_- = \dim(|C| + C).$$

Then  $n_{\pm} > 0$  by indefiniteness of  $C$ . The triple  $\{n_+, n_-, n_0\}$  with  $n_0 \stackrel{\text{def}}{=} \dim \ker(C)$  is usually called the *inertia* of selfadjoint  $C$ . Fix an invertible operator  $V \in \mathcal{B}(\mathcal{H})$  such that with respect to decompositions (5.2) and (5.3)

$$V = \begin{bmatrix} V_{11} & 0 \\ 0 & I_0 \end{bmatrix} \quad \text{and} \quad C = V^* \cdot \begin{bmatrix} I_+ & 0 & 0 \\ 0 & -I_- & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot V, \quad (5.4)$$

where  $I_0$  and  $I_{\pm}$  are the identity matrices of order  $n_0$  and  $n_{\pm}$  respectively.



**Theorem 5.** *Let  $\dim(\mathcal{H}) < \infty$  and let  $A, B > 0$ , and let  $C$  and  $V$  be as in (5.4). Then every invertible extreme point  $X$  of  $\Delta(A, B)$  is uniquely written in the form*

$$X = \frac{1}{2}(A + B) - V^* \cdot \begin{bmatrix} D(K) & 0 \\ 0 & 0 \end{bmatrix} \cdot V, \quad (5.5)$$

where  $K$  is an  $n_- \times n_+$  matrix with  $K^*K < I_+$  and

$$D(K) \stackrel{\text{def}}{=} \begin{bmatrix} (I_+ + K^*K)(I_+ - K^*K)^{-1} & 2(I_+ - K^*K)^{-1}K^* \\ 2K(I_+ - K^*K)^{-1} & (I_- + KK^*)(I_- - KK^*)^{-1} \end{bmatrix}. \quad (5.6)$$

Conversely each  $n_- \times n_+$  matrix  $K$  with  $K^*K < I_+$  gives rise to an invertible extreme point  $X$  of  $\Delta(A, B)$  by (5.5) and (5.6).

## 6. Construction

Let  $A_j \geq 0$  ( $j = 1, 2, \dots, m$ ). Obviously 0 is an extreme point, which is  $\leq X$  for all  $X \in \Delta(A_1, A_2, \dots, A_m)$ . In this section along an idea of [Ao 93] we shall present an algorithm of obtaining an extreme point  $\tilde{X}$  such that  $X \leq \tilde{X}$  for given  $X \in \Delta(A_1, A_2, \dots, A_m)$ .

**Theorem 6.** *Let  $X \in \Delta(A_1, \dots, A_m)$ . Starting with  $X_0 \stackrel{\text{def}}{=} X$ , define successively*

$$X_{k+1} \stackrel{\text{def}}{=} X_k + \prod_{j=1}^k (A_j - X_k) \quad (k = 1, 2, \dots). \quad (6.1)$$

Then  $\{X_k \mid k = 1, 2, \dots\}$  is an increasing sequence in  $\Delta(A_1, \dots, A_m)$  and its (strong) limit  $X_\infty \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} X_k$  is an extreme point of  $\Delta(A_1, \dots, A_m)$  such that

$$X \leq X_\infty \quad \text{and} \quad \prod_{j=1}^m (A_j - X_\infty) = 0. \quad (6.2)$$

Remark that the algorithm in Theorem 6 produces all extreme points  $X$  of  $\Delta(A_1, A_2, \dots, A_m)$  such that

$$\prod_{j=1}^m (A_j - X) = 0,$$

because for such  $X$  it is immediate to see that all  $X_k$  coincide with  $X$  for all  $k$ , and  $X_\infty = X$ .

**Theorem 7.** Let  $\dim(\mathcal{H}) < \infty$  and  $X \in \Delta(A_1, A_2, \dots, A_m)$ . Starting with  $X_0 \stackrel{\text{def}}{=} X$ , define successively

$$X_{k+1} \stackrel{\text{def}}{=} X_k + \prod_{j=1}^k : ([X]A_j - X_k) \quad (k = 1, 2, \dots). \quad (6.4)$$

Then  $\{X_k \mid k = 1, 2, \dots\}$  is an increasing sequence in  $\Delta(A_1, \dots, A_m)$  and its limit  $X_\infty \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} X_k$  is an extreme point of  $\Delta(A_1, \dots, A_m)$  such that

$$X \leq X_\infty = [X]X_\infty \quad \text{and} \quad \prod_{j=1}^m : ([X_\infty]A_j - X_\infty) = 0. \quad (6.5)$$

Conversely all extreme points are obtained in this way.

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